From the pseudo-range overbounding to the integrity risk overbounding

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BIOGRAPHY

Igor Nikiforov received his M.S. degree in automatic control from the Moscow Physical - Technical Institute in 1974, and the Ph.D. in automatic control from the Institute of Control Sciences (ICS), Russian Academy of Science, Moscow, Russia in 1981. He is a professor at the University of Technology of Troyes, Charles Delaunay Institute, UMR 6281 CNRS, France. He is currently active or has demonstrated activity in the past in the following areas: sequential change detection and isolation; sequential analysis; statistical decision with nuisance parameters; parametric statistical models; statistical fault detection and isolation; multivariable control theory; statistical process control; seismic signal processing; cyber(-physical) security; navigation system integrity monitoring.

ABSTRACT

The advantages of a navigation system that can monitor its own integrity are obvious. All GNSS integrity monitoring methods may be broadly divided into two classes: “active integrity methods” and “passive integrity methods”. The active integrity methods, like ARAIM, are very efficient in the case of unbounded GNSS channel degradations.

However, nowadays even bounded degradations of the pseudo-range measurements lead to an unacceptable additional bias of the users fix, due to reduced alarm limits. In such a case, the solution consists in the passive integrity method, which is called “overbounding”. The overbounding approach allows all pseudo-range measurements without rejection. The idea of the approach is to define conservative bounds (overbounds) for the cumulative distribution function (CDF) of pseudo-range errors in order to get a conservative bound for the integrity risk.

The known methods of overbounding are applicable to the calculation of conservative bounds for the distribution of a linear combination of several random variables. Hence, these methods can be used to calculate the instantaneous integrity risk overbounding for the vertical positioning but they cannot be used to calculate the instantaneous integrity risk overbounding for the horizontal positioning. This is because the horizontal positioning error is a nonlinear combination of several pseudo-range errors.

Moreover, the MOPS for GPS/Galileo requires calculating the integrity risk “per a given period of time” (e.g., “per approach” or “per hour”). Because the pseudo-range errors are strongly auto-correlated, the passage from the instantaneous integrity risk overbounding to the integrity risk overbounding per a given period of time is not trivial.

The original contribution of this paper is twofold. First, we propose two new conservative bounds for the integrity risk in the horizontal plane by using one or two Gaussian PDFs with an inflation coefficient. Second, we calculate the impact of the pseudo-range errors autocorrelation on the conservative bounds for the vertical/horizontal integrity risks.

1. INTRODUCTION AND MOTIVATION

The GNSS integrity monitoring methods can be broadly divided into two classes: “active integrity methods” and “passive integrity methods”. For example, the Advanced Receiver Autonomous Integrity Monitoring (ARAIM) with Fault Detection/Exclusion (FDE) functions belongs to the class of active integrity methods. If an unbounded additional pseudo-range bias in one (or two) GNSS channel(s) occurs at an unknown time then the outputs of the Least Squares (LS) algorithm are not an optimal solution. Such unbounded pseudo-range bias leads to an unbounded additional bias of the user’s fix, which is clearly very undesirable. In this case, the only solution to preserve a high constant integrity level of GNSS positioning is to use the active integrity methods, like ARAIM.

However, nowadays another type of problem arises due to reduced horizontal/vertical alarm limits: some bounded degradations (additional biases and/or CDF shape deformation) of (maybe several simultaneous) pseudo-range measurement(s) lead to an unacceptable additional bias of the user’s fix. Unfortunately, the probabilities of false alarm, missed detection and false isolation of ARAIM FDE algorithms (even in the case of optimal statistical tests) for such bounded degradations do not satisfy the advanced MOPS for GPS/Galileo for some safety-critical mode of flight.

In that case, a reasonable solution to the problem of integrity monitoring consists in the passive integrity method, based on pseudo-ranges “overbounding”. The overbounding approach allows all pseudo-range measurements without rejection. The concept of overbounding suggests to define conservative bounds (overbounds) for the cumulative distribution function (CDF) or probability density function (PDF) of pseudo-range errors and to get a conservative bound for the integrity risk by using such overbounds for the pseudo-range errors.

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The rest of the paper is organized as follow. The pseudo-range measurement equations for single/double frequencies are introduced in Section 2. The navigation solution and the expression of the horizontal and vertical positioning errors as functions of the pseudo-range errors are given in Section 3. Next, the problem of overbounding is stated and the original contribution is defined in Section 4. A short critical analysis of the overbounding methods available in the literature is given in Section 5. The overbounding for horizontal/vertical instantaneous integrity risk is calculated in Section 6. The overbounding for horizontal/vertical "per-given-period-of-time" integrity risk is calculated in Section 7. Here, we study the impact of the pseudo-range errors autocorrelation on the "per-given-period-of-time" integrity risk. Three numerical examples of overbounding are given in Section 8. Finally, some conclusions are drawn in Section 9.

2. MEASUREMENT EQUATIONS

The navigation solution is based upon accurate measuring the distance (*range*) from several satellites with known locations to a user (vehicle). Let us assume that there are m satellites located in three-space (ECEF coordinates) at the known positions $X_i = (x_i, y_i, z_i)^T, i = 1, \dots, m$, and a user at $X_u = (x, y, z)^T$. The *pseudo-ranges* r_i ($i = 1, \dots, m$) from the navigation satellites to the user can be written as

$$r_i = d_i(x, y, z) + c\, t_r + \xi_i, \tag{1}$$

where $d_i(x, y, z) = \|X_i - X_u\|_2$ is the true distance from the i-th satellite to the user, t_r is a user clock bias, $c \simeq 2.9979 \cdot 10^8$m/s is the speed of light and ξ_i is an additive error of the pseudo-range r_i. Let us define the vector $\xi = (\xi_1, \dots, \xi_m)^T$ of additive pseudo-range errors at the user's position, the vector $B = (b_1, \dots, b_m)^T$, $b_i = \mathbb{E}(\xi_i)$, of nominal pseudo-range biases (unknown but bounded mean error) and the diagonal variance-covariance matrix $\Sigma = \mathrm{diag}\left\{\sigma_1^2, \dots, \sigma_m^2\right\}$, where the variances $\sigma_i^2 = \mathrm{Var}(\xi_i)$, $i = 1, \dots, m$, are defined as functions of several parameters, for instance, elevation angles.

Let us suppose that the distances from the satellites to the user (see equation (1)) are measured by using two frequencies f_1 and f_2 (for example L1 and E5$_a$) :

$$\begin{aligned} r_{1,\,i} &= d_i(x, y, z) + c\, t_r + \xi_{1,\,i} \quad \text{frequency } f_1 \\ r_{2,\,i} &= d_i(x, y, z) + c\, t_r + \xi_{2,\,i} \quad \text{frequency } f_2 \end{aligned} \tag{2}$$

The dual-frequency (corrected) pseudo-range $r_{1-2,\,i}$ is a linear combination of the pseudo-ranges $r_{1,\,i}$ and $r_{2,\,i}$ measured with the frequencies f_1 and f_2 [1] :

$$\begin{aligned} r_{1-2,\,i} &= \frac{\gamma}{\gamma - 1} r_{1,\,i} - \frac{1}{\gamma - 1} r_{2,\,i} \\ &= d_i(x, y, z) + c\, t_r + \xi_{1-2,\,i}, \end{aligned}$$

where $\xi_{1-2,\,i} = [\gamma/(\gamma - 1)]\xi_{1,\,i} - [1/(\gamma - 1)]\xi_{2,\,i}$, $\gamma = (f_1/f_2)^2 > 1$. By analogy with the mono-frequency signal, let us define the vector ξ_{1-2} of dual-frequency pseudo-range errors, the vector B_{1-2} of nominal biases and the diagonal variance-covariance matrix Σ_{1-2}.

3. NAVIGATION SOLUTION

Let us first linearize the measurement equations (1) and (2) around a working point $X_{u\,0} = (x_0, y_0, z_0)^T$. In fact, the mathematical background is essentially the same and the only difference is the vector of nominal bounded biases and the matrix of variances of pseudo-range errors. Let us consider r_i as a function of x, y, z and t_r :

$$(X_u, t_r) \mapsto r_i = d_i(x, y, z) + ct_r + \xi_i \tag{3}$$

and let us introduce the following vectors: $R = (r_1, \dots, r_m)^T$, $D = (d_1, \dots, d_m)^T$ and $X = (X_u^T, ct_r)^T$.

For the sake of simplicity, we consider in the rest of the paper that the ECEF coordinates is transformed to the local East, North, Up (ENU) coordinates. By linearizing the pseudo-range equation with respect to the vector X_u around the working point $X_{u\,0} = (x_0, y_0, z_0)^T$, we get the measurement equation

$$Y = R - D_0 \simeq H(X - X_0) + \xi, \tag{4}$$

where $Y = (y_1, \dots, y_m)^T$, $D_0 = (d_{1\,0}, \dots, d_{m\,0})^T$, $d_{i\,0} = \|X_i - X_{u\,0}\|_2$, $X_0 = (X_{u\,0}^T, 0)^T$ and $H = \frac{\partial R}{\partial X}\Big|_{X=X_0}$ is a Jacobian matrix of size $(m \times 4)$. In the case of dual-frequency measurements, R is replaced by $R_{1-2} = (r_{1-2,\,1}, \dots, r_{1-2,\,m})^T$ and ξ by ξ_{1-2} in (4). The LS method :

$$\widehat{X} = X_0 + A(R - D_0), \ A = (H^T\Sigma^{-1}H)^{-1}H^T\Sigma^{-1}. \tag{5}$$

is the best linear unbiased estimator of X under assumption that $m \geq 5$, $B = \mathbb{E}(\xi) = 0$ and $\mathrm{cov}(\xi) = \Sigma$ (in the dual-frequency case, $\Sigma = \Sigma_{1-2}$) is known, see details in [2], [3]. Moreover, it follows from the theorem of Rao-Cramer [2] that the LS method reaches the Cramér-Rao lower bound in the class of unbiased estimator of X : if $\xi \sim \mathcal{N}(0, \Sigma)$ then

$$\mathrm{cov}_X\left(\widetilde{X}\right) \geq \mathrm{cov}_X\left(\widehat{X}\right) = \left(H^T\Sigma^{-1}H\right)^{-1}, \tag{6}$$

where $\widetilde{X}$ is any unbiased estimator of X and $\widehat{X}$ is the LS estimator and simultaneously a lower bound in the class of unbiased estimators.

To calculate a conservative bound for the integrity risk, it is necessary to express the horizontal and vertical positioning errors as functions of the pseudo-range errors $\xi_1, \dots, \xi_m$. As it follows from (5), the vector of positioning errors $\widehat{X} - X$ (in ENU coordinates) is a linear combination of the pseudo-range errors $\xi_1, \dots, \xi_m$

$$\widehat{X} - X = A\xi. \tag{7}$$

4. PROBLEM STATEMENT AND ORIGINAL CONTRIBUTION

If the distribution of the pseudo-range errors ξ is Gaussian $\mathcal{N}(B, \Sigma)$, where $B = \mathbb{E}(\xi) = 0$ and the variance-covariance matrix $\mathrm{cov}(\xi) = \Sigma$ is known, the estimation $\widehat{X}$ realizes the smallest possible ellipsoid of errors $\widehat{X} - X$. But the LS estimation given by (5) is optimal only theoretically, under the above-mentioned conditions.

We are interested what happens if $B = \mathbb{E}(\xi) \neq 0$, the variance-covariance matrix Σ is only partially known and, moreover, the distribution $F_\xi(x)$ of the pseudo-range errors ξ (or ξ_{1-2}) is unknown but only conservative bounds (overbounds) for the CDF $F_\xi(x)$ and/or PDF $f_\xi(x) = dF_\xi(x)/dx$ of pseudo-range errors are available. The question is how to get conservative bounds for the integrity risk in such situation.

4.1 Instantaneous integrity risk overbounding

The instantaneous (per GNSS epoch) integrity risk for the horizontal and vertical positioning is defined by the following probabilities

$$\mathbb{P}\left(\|Q_h\|_2 \geq \text{HAL}\right), \tag{8}$$

where $Q_h = (\widehat{x} - x, \widehat{y} - y)^T$ and HAL means the Horizontal Alarm Level, and

$$\mathbb{P}\left(|Q_v| \geq \text{VAL}\right), \tag{9}$$

where $Q_v = \widehat{z} - z$ and VAL means the Vertical Alarm Level.

Hence, the first goal of this paper is to calculate conservative upper bounds for the horizontal and vertical positioning integrity risk, defined in (8) and (9), as functions of the overbounds for the CDF $F_\xi(x)$ and/or PDF $f_\xi(x)$.

4.2 "Per a given period of time" integrity risk overbounding

Nevertheless, in reality, the MOPS for GPS/Galileo require calculating the integrity risk per a given period of time (e.g., "per approach" or "per hour"). Because the pseudo-range errors are strongly auto-correlated, the passage from the instantaneous risk to the risk per a given period of time and its overbounding are not trivial. Let us define two random sequences of positioning errors :

$$\{Q_{h,n}\}_{n\geq 1} \quad \text{and} \quad \{Q_{v,n}\}_{n\geq 1}, \tag{10}$$

where $Q_{h,n} = (\widehat{x}_n - x_n, \widehat{y}_n - y_n)^T$, $Q_{v,n} = \widehat{z}_n - z_n$, $n = 1, 2, 3, \ldots$ is the current number of time step (GNSS epoch) and the following stopping times N :

$$N_h = \inf\{n \geq 1 : \|Q_{h,n}\|_2 \geq \text{HAL}\}, \tag{11}$$

$$N_v = \inf\{n \geq 1 : |Q_{v,n}| \geq \text{VAL}\}. \tag{12}$$

Therefore, the horizontal and vertical integrity risk per a given period of time T is defined as the following conditional probabilities

$$\mathbb{P}\left(N_h \leq T \middle| \|Q_{h,0}\|_2 < \text{HAL}\right) \tag{13}$$

and

$$\mathbb{P}\left(N_v \leq T \middle| |Q_{v,0}| < \text{VAL}\right). \tag{14}$$

provided that the starting point $Q_{h,0}$ (resp. $Q_{v,0}$) satisfies the condition $\|Q_{h,0}\|_2 < \text{HAL}$ (resp. $|Q_{v,0}| < \text{VAL}$).

The second goal of this paper is to calculate conservative upper bounds for the horizontal and vertical positioning integrity risk per a given period of time defined in (13) and (14) as functions of the pseudo-range autocorrelations, the overbounds for the CDF $F_\xi(x)$ and/or PDF $f_\xi(x)$.

4.3 The original contribution of this paper

First, we propose two new methods of overbounding in the horizontal plane. It is assumed that the degradation of the pseudo-range error with the PDF $f_{\xi_i}(x)$ leads to two different phenomena : *i)* the deformation of the PDF $f_{\xi_i}(x)$ shape; *ii)* the emergence of an additional (unknown but bounded) bias b_i in the pseudo-range errors ξ_i. To cover this degradation, it is proposed to use one or two Gaussian PDFs with an inflation coefficient. These conservative Gaussian bounds for pseudo-range errors $\xi_1, \ldots, \xi_m$ are transformed into the conservative bound for the instantaneous integrity risk (8) in the horizontal plane. Simple analytical and numerical expressions are proposed for such a conservative bound.

Second original contribution consists in establishing the impact of the pseudo-range errors autocorrelation on the conservative bound for the vertical/horizontal integrity risks per a given period of time T. The proposed solution is reduced to the first-passage-problem for the autoregressive first order model AR(1). In the case of horizontal positioning, our goal is to calculate a conservative bound for the probability $\mathbb{P}\left(N_h \leq T \middle| \|Q_{h,0}\|_2 < \text{HAL}\right)$ that the vector AR(1) process reaches the circular absorbing boundary of radius HAL during a given period T (expressed as the number of time units, i.e., GNSS epochs that have elapsed since a specified epoch). In the case of vertical positioning, our goal is to calculate a conservative bound for the probability $\mathbb{P}\left(N_v \leq T \middle| |Q_{v,0}| < \text{VAL}\right)$ that the scalar AR(1) process reaches the absorbing boundary $\pm$VAL during a given period of time T.

The proposed contributions represent a mathematical background to formalize the vertical and horizontal integrity risk overbounding in exact terms following the requirements of the MOPS for GPS/Galileo.

5. KNOWN METHODS OF OVERBOUNDING

The goal of this section is to recall the known methods of overbounding. The section is concluded by formulating some open problems.

Three main methods of overbounding for scalar random variables are available in the literature :

- the method of "Single CDF overbounding" (see DeCleene [4]) applicable to symmetric and unimodal distributions;
- the method of "Paired CDF Overbounding" (see Rife, Pullen, Enge, and Pervan [5], [6], [7];
- the method of "Excess-Mass CDF overbounding" (see Rife, Walter and Blanch [8], [9], [10].

5.1 Single CDF overbounding

Definition 1: The single CDF overbounding is defined as follows [4]:

$$F_{o,\xi}(x) \geq F_\xi(x) \;\; \forall x \leq 0 \;\; \text{et} \;\; F_{o,\xi}(x) \leq F_\xi(x) \;\; \forall x > 0,$$

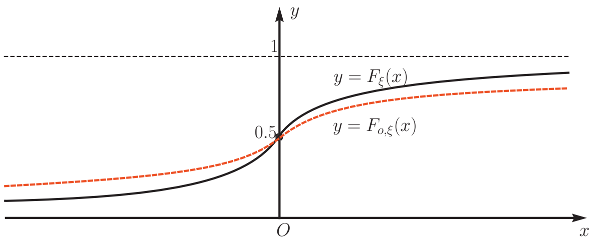

Fig. 1. Single CDF overbounding.

where $F_\xi(x)$ is the CDF of the random variable ξ, $F_{o,\xi}(x)$ is the bound for the CDF $F_\xi(x)$.
This type of overbounding is shown in Figure 1.

Proposition 1 (DeCleene [4]): Let us consider two independent random variables ξ and η. Their CDF functions are denoted as $F_\xi(x)$ and $F_\eta(x)$ and their functions of overbounding are denoted as $F_{o,\xi}(x)$ and $F_{o,\eta}(x)$ (see Definition 1). It is considered that the distributions are zero-mean, unimodal and symmetric. Then

$$F_{o,\xi+\eta}(x) \geq F_{\xi+\eta}(x)\ \forall x \leq 0$$

and

$$F_{o,\xi+\eta}(x) \leq F_{\xi+\eta}(x)\ \forall x > 0$$

where $F_{o,\xi+\eta}(x) = (F_{o,\xi} * F_{o,\eta})(x)$ and $F_{\xi+\eta}(x) = (F_\xi * F_\eta)(x)$ are the convolutions of the CDF.

5.2 Paired CDF Overbounding

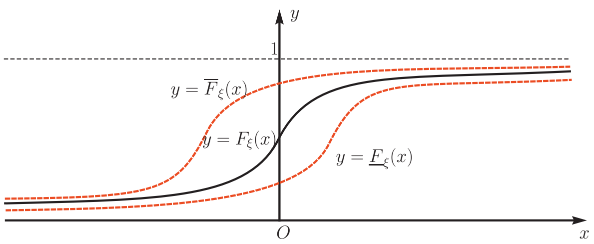

Fig. 2. Paired CDF overbounding.

Definition 2: The paired CDF overbounding is defined as follows (see [5], [6]) :

$$\underline{F}_\xi(x) \leq F_\xi(x) \leq \overline{F}_\xi(x) \quad \forall x \in \mathbb{R}$$

This overbounding is shown in Figure 2.

Theorem 1 (J. Rife, S. Pullen, P. Enge et B. Pervan [5], [6]): Let us consider two independent random variables ξ and η with the CDF $F_\xi(x)$ and $F_\eta(x)$ and the functions of overbounding $\underline{F}_\xi(x), \overline{F}_\xi(x)$ and $\underline{F}_\eta(x), \overline{F}_\eta(x)$, respectively. Then

$$\underline{F}_{\xi+\eta}(x) \leq F_{\xi+\eta}(x) \leq \overline{F}_{\xi+\eta}(x)\ \ \forall x \in \mathbb{R}$$

where $\underline{F}_{\xi+\eta}(x) = (\underline{F}_\xi * \underline{F}_\eta)(x)$, $\overline{F}_{\xi+\eta}(x) = (\overline{F}_\xi * \overline{F}_\eta)(x)$ and $F_{\xi+\eta}(x) = (F_\xi * F_\eta)(x)$.

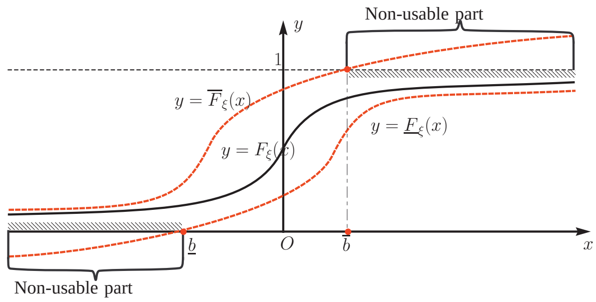

Fig. 3. Excess-Mass CDF overbounding.

5.3 Excess-Mass CDF overbounding

The "Excess-Mass CDF overbounding" can be interpreted as the extension of the paired CDF overbounding, where the natural constraints : $0 \leq \underline{F}_\xi(x) \leq \overline{F}_\xi(x) \leq 1$ are not respected and the bounds $x \mapsto \underline{F}_\xi(x)$ et $x \mapsto \overline{F}_\xi(x)$ can be negative or positive but greater than 1. This situation is illustrated in Figure 3. The idea of this approach is to get a more flexible bounds $x \mapsto \underline{F}_\xi(x)$ and $x \mapsto \overline{F}_\xi(x)$. If the bounds are based on the Gaussian distribution [8], [9], [10], the overbounding equations are :

$$\underline{F}_\xi(x; K, \theta, \gamma) = K \int_{-\infty}^{x} f(u; \theta, \gamma\sigma)du + (1 - K) \quad (15)$$

and

$$\overline{F}_\xi(x; K, \theta, \gamma) = K \int_{-\infty}^{x} f(u; -\theta, \gamma\sigma)du, \quad (16)$$

where $f(x; \theta, \sigma)$ is the PDF of the Gaussian law $\mathcal{N}(\theta, \sigma^2)$. The convolutions defined in Theorem 1 are also applicable to the bounds defined by the "Excess-Mass CDF overbounding".

5.4 Critical analysis and open problems

- The weak point of the single CDF overbounding is the necessity of very strong assumptions : the distributions of the pseudo-range errors should be zero-mean, unimodal and symmetric. It seems that so strong assumptions are not compatible with the biases b_i in the pseudo-range errors ξ_i and the deformation of the PDF $f_{\xi_i}(x)$ shape.
- The strong point of the paired CDF overbounding is that the distributions can be arbitrary (even those that are not zero-mean, unimodal and symmetric). This method seems to be more realistic, because the biases b_i in the pseudo-range errors ξ_i and the deformation of the PDF $f_{\xi_i}(x)$ shape are compatible with the paired CDF overbounding.
- The excess-mass CDF overbounding is a potentially interesting approach but it can produce very conservative bounds for the CDF.

The paired CDF overbounding (Paired Overbound Theorem 1) is well-adapted to the situation where the estimation is a linear combination of several independent pseudo-range errors $\xi_1, \ldots, \xi_m$. As it follows from (7), the LS estimation error on the vertical axis is calculated as a weighted sum of the pseudo-range random errors $\xi_i \sim F_{\xi_i}$, where F_{ξ_i} denotes the CDF of the pseudo-range ξ_i :

$$Q_v = \widehat{z} - z = \sum_{i=1}^{m} a_{3,i}\xi_i \qquad (17)$$

where $a_{j,i}$ is the (j,i)-th entry of the matrix A. Hence, Theorem 1 can be easily applied to the vertical risk overbounding.

On the contrary, in the horizontal risk overbounding, the radial error r is a nonlinear function of several independent pseudo-range errors $\xi_1, \ldots, \xi_m$.

$$r = \|Q_h\|_2 = \sqrt{(\widehat{x}-x)^2 + (\widehat{y}-y)^2}, \qquad (18)$$

where $Q_h = (\widehat{x}-x, \widehat{y}-y)^T$ with

$$\widehat{x} - x = \sum_{i=1}^{m} a_{1,i}\xi_i, \;\; \widehat{y} - y = \sum_{i=1}^{m} a_{2,i}\xi_i \qquad (19)$$

and $a_{j,i}$ is the (j,i)-th entry of the matrix A. Hence, Theorem 1 is not applicable to the horizontal risk overbounding.

All three above-mentioned methods of overbounding are applicable only for the calculation of the conservative bounds for the instantaneous vertical risk.

Finally, it can be concluded that there are no methods available in the literature

- for calculation of the conservative bounds for the instantaneous horizontal risk;
- for calculation of the conservative bounds for the vertical/horizontal integrity risks per a given period of time T.

6. CONSERVATIVE BOUNDS FOR THE INSTANTANEOUS INTEGRITY RISKS

The main goal of this section is to find conservative bounds for the horizontal instantaneous integrity risk. This problem is divided in two steps : *i)* bounding the impact of the pseudo-range biases on the conservative bounds for Gaussian distributions; *ii)* bounding the impact of the PDF $f_{\xi_i}(x)$ shape deformation on the conservative bounds for an arbitrary distribution. Finally, the conservative bound for the vertical instantaneous integrity risk will be briefly discussed by using Theorem 1.

6.1 Conservative bounds for the horizontal instantaneous integrity risk (Gaussian distribution)

As it follows from Section 4, a more realistic working hypothesis includes an additional bounded bias b_i in the pseudo-range measurement (1), (2). Let us consider that the errors ξ_i are distributed following the Gaussian distribution $\xi_i \sim \mathcal{N}(b_i, \sigma_i^2)$ and that the absolute value of the bias b_i of the pseudo-range measurement r_i is upper bounded by $\overline{b}_i$. Hence

$$-\overline{b}_i \leq b_i \leq \overline{b}_i, \quad i = 1, \ldots, m.$$

This last condition can be interpreted as the "Paired CDF overbounding" in the case where the class of possible distributions of the pseudo-range errors ξ_i is restricted to a family of Gaussian distributions with bounded means. The functions of overbounding are given by (see Definition 2)

$$\underline{F}_{i,\xi}(x) = \mathcal{N}(\overline{b}_i, \sigma_i^2) \quad \text{and} \quad \overline{F}_{i,\xi}(x) = \mathcal{N}(-\overline{b}_i, \sigma_i^2).$$

Let us recall some useful results on the probability calculation for Gaussian quadratic forms. Suppose that $X \sim \mathcal{N}(\theta, \Sigma)$, $\Sigma \in \mathcal{M}_{\ell \times \ell}$ is the variance-covariance matrix of X and $\theta \in \mathbb{R}^\ell$ is the vector of means of X. Our goal is to calculate the probability of the event $\|X\|_2 \geq h$ by using the function $F_\ell(y, \Lambda, \omega)$

$$\mathbb{P}\left(\|X\|_2 \geq h\right) = 1 - F_\ell\left(h^2, \Lambda, \omega\right). \qquad (20)$$

This function

$$F_\ell(y, \Lambda, \omega) = (2\pi)^{-\frac{\ell}{2}} \underbrace{\int \cdots \int}_{D=\{(W-\omega)^T\Lambda(W-\omega)\leq y\}} \exp\left\{-\frac{1}{2}\|W\|_2^2\right\} dW,$$

where $W \in \mathbb{R}^\ell$ denotes the support of the random vector $U^T\xi \sim \mathcal{N}(0, I_\ell)$ and $\omega = -\Lambda^{-\frac{1}{2}}U^T\theta$, is well-known in the statistical literature. A numerical method of its calculation is given in [11].

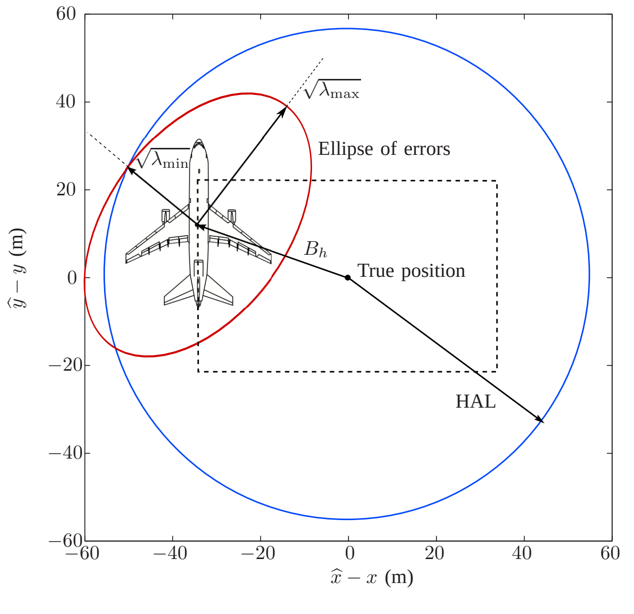

Fig. 4. The vector of systematic errors B_h and the orientation of the error ellipse. The major radius and minor radius are denoted by $\sqrt{\lambda_{\max}}$ and $\sqrt{\lambda_{\min}}$, respectively.

Let us assume that $X = Q_h$, $\theta = B_h$, $\ell = 2$ and $h = \text{HAL}$. The analysis of the function $F_2(y, \Lambda, \omega)$ shows that there are two factors determining the probability (20) :

- the vector of systematic horizontal errors $B_h \in \mathbb{R}^2$;
- the orientation φ of the ellipse with respect to the vector B_h.

This situation is illustrated by Figure 4.

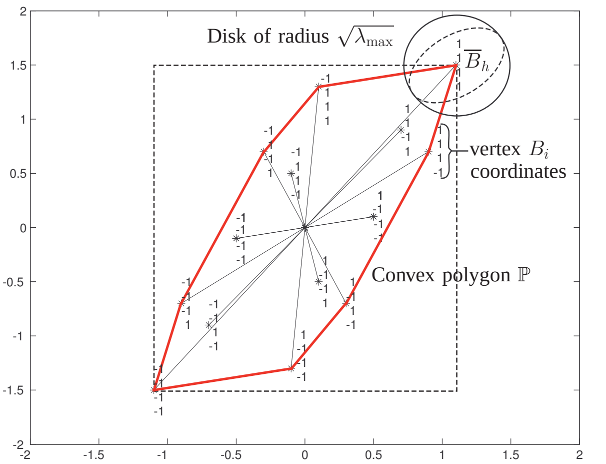

Fig. 5. The convex polygon $\mathbb{P}$ obtained by a linear mapping (defined by the matrix A_h) of the set $\mathbb{B}$ onto the set $\mathbb{P}$.

By using (7), the vector of systematic errors B_h can be expressed as a linear function of the vector of pseudo-range biases B :

$$B_h = A_h B, \quad A_h = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ a_{2,1} & \cdots & a_{2,m} \end{pmatrix}.$$

Let us define the following hyperrectangle $\mathbb{B} = \left\{X \in \mathbb{R}^m | x_i \in [-\overline{b}_i, \overline{b}_i],\ i = 1, \ldots, m\right\}$ and a linear mapping (defined by the matrix A_h) of the set $\mathbb{B}$ onto the set $\mathbb{P}$. The set $\mathbb{P}$ is a convex polygon. This convex polygon is shown in Figure 5 for a toy example of four visible satellites, i.e., $m = 4$ and $\overline{b}_i = 1,\ i = 1, \ldots, 4$.

Because the vector of pseudo-range biases B is unknown but bounded, i.e., $B \in \mathbb{B}$, the problem of overbounding in the horizontal plane is reduced to the maximization of the probability to lose integrity (i.e., instantaneous risk of integrity). In such a way, we get a conservative (worst-case) estimation of the instantaneous risk of integrity. The maximization of this probability is equivalent to the minimization of the function $F_2(\ldots)$:

$$\max_{B\in\mathbb{B},\varphi} \mathbb{P}\left(\|Q_h\|_2 \geq \text{HAL}\right) = 1 - \min_{B\in\mathbb{B},\varphi} F_2\left(\text{HAL}^2, \Lambda, -\Lambda^{-\frac{1}{2}} U^T A_h B\right), \quad (21)$$

where the horizontal errors $Q_h = (\widehat{x} - x, \widehat{y} - y)^T$ (in ENU coordinates) are defined in (19), the matrices Λ and U are calculated by using the variance-covariance matrix $\Sigma = \text{cov}\left(Q_h\right) = U \Lambda U^T$.

The calculation of the probability $\mathbb{P}_{B_h,\varphi}\left(\|X\|_2 \geq \text{HAL}\right)$, defined by (20), for different values B_h and for different orientations φ of the ellipse, represents a heavy computational burden (especially for on-board computer). To reduce computational burden, the error ellipse can be overestimated by a disk of the radius $\sqrt{\lambda_{\max}}$, where $\lambda_{\max} = \max\{\lambda_1, \lambda_2\}$ and λ_1, λ_2 are eigenvalues of the matrix Σ. In this case, the probability to lose integrity is overbounded in the following manner

$$\mathbb{P}_{B_h,\varphi}\left(\|X\|_2 \geq \text{HAL}\right) \leq 1 - F_2\left(\text{HAL}^2, \overline{\Lambda}, \omega\right), \quad (22)$$

where $\overline{\Lambda} = \text{diag}\left\{\lambda_{\max}, \lambda_{\max}\right\}$ and $\omega = -\overline{\Lambda}^{-\frac{1}{2}} B_h$. This overbounding simplifies the problem of minimization (21). By putting together (21) and (22), we get

$$\max_{B\in\mathbb{B},\varphi} \mathbb{P}\left(\|Q_h\|_2 \geq \text{HAL}\right) \leq 1 - \min_{B\in\mathbb{B}} F_2\left(\text{HAL}^2, \overline{\Lambda}, -\overline{\Lambda}^{-\frac{1}{2}} A_h B\right), \quad (23)$$

where $\overline{\Lambda} = \text{diag}\left\{\lambda_{\max}, \lambda_{\max}\right\}$. In this last case, the function $F_2(\ldots)$ depends on the norm $\|A_h B\|_2$ and this function $\|A_h B\|_2 \mapsto F_2(\ldots, \|A_h B\|_2)$ is monotone decreasing for any given HAL and $\overline{\Lambda}$. Therefore, the upper bound for the horizontal integrity risk (defined in the right-hand side of (23)) is a monotone increasing function of the Euclidean norm $\|A_h B\|_2$. Finally, the problem of conservative estimation of the horizontal integrity risk (21) – (23) is reduced to the maximisation of the Euclidean norm over the convex polygon $\mathbb{P}$. The maximum is reached in a vertex of the convex polygon $\mathbb{P}$. Hence, the overbounding for the instantaneous risk of integrity is given by

$$\max_{B\in\mathbb{B},\varphi} \mathbb{P}(\|Q_h\|_2 \geq \text{HAL}) \leq 1 - F_2\left(\text{HAL}^2, \overline{\Lambda}, -\overline{\Lambda}^{-\frac{1}{2}} \overline{B}_h\right), \quad (24)$$

where $\overline{B}_h = A_h B_j$, $j = \arg \max\limits_{i=1,\ldots,2^m} \{\|A_h B_i\|_2\}$, and B_i is a vertex of the hyperrectangle $\mathbb{B}$, $i = 1, \ldots, 2^m$.

As it follows from (24), the maximization of the Euclidean norm $\|A_h B\|_2$ over the set of 2^m vertices of the hyperrectangle $\mathbb{B}$ can be simplified by using the vector B_h^* defined as follows

$$B_h^* = \begin{pmatrix} |a_{1,1}|\overline{b}_1 & + & \cdots & + & |a_{1,n}|\overline{b}_m \\ |a_{2,1}|\overline{b}_1 & + & \cdots & + & |a_{2,n}|\overline{b}_m \end{pmatrix}. \quad (25)$$

instead of $\overline{B}_h$. Hence, the following (more conservative) overbounding for the horizontal integrity risk is given by

$$\max_{B\in\mathbb{B},\varphi} \mathbb{P}(\|Q_h\|_2 \geq \text{HAL}) \leq 1 - F_2\left(\text{HAL}^2, \overline{\Lambda}, -\overline{\Lambda}^{-\frac{1}{2}} \overline{B}_h\right) \leq 1 - F_2\left(\text{HAL}^2, \overline{\Lambda}, -\overline{\Lambda}^{-\frac{1}{2}} B_h^*\right). \quad (26)$$

6.2 Conservative bounds for the horizontal instantaneous integrity risk (arbitrary distribution)

1) Excess-mass overbounding by using two Gaussian PDFs: Let us consider that the true PDF $f_{i,\xi}(x)$ of the pseudo-range errors ξ_i can be upper bounded by using two PDFs $f_{b_i}(x)$ and $f_{-b_i}(x)$ of two Gaussian laws $\mathcal{N}(b_i, \sigma_i^2)$ and $\mathcal{N}(-b_i, \sigma_i^2)$:

$$f_{i,\xi}(x) \leq c_i [f_{b_i}(x) + f_{-b_i}(x)] \text{ for } x \in \mathbb{R} \text{ and } i = 1, \ldots, m,$$

where c_i is the inflation coefficient and $b_i > 0$. This method of overbounding, illustrated in Figure 6, can be interpreted as the method of "Excess-Mass PDF overbounding" proposed in [8].

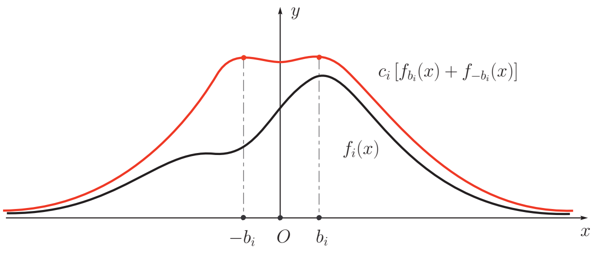

Fig. 6. Overbounding of the pseudo-range error $\widetilde{\xi}_i$ PDF by using two Gaussian PDFs.

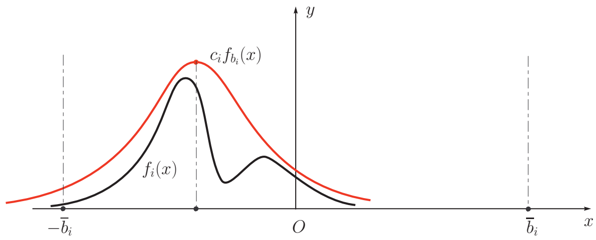

Fig. 7. Overbounding of the pseudo-range error $\widetilde{\xi}_i$ PDF by using a single Gaussian PDF with adapted mean.

To estimate the horizontal integrity risk, it is necessary to calculate the following multiple integral :

$$\mathbb{P}(\|Q_h\|_2 \geq \text{HAL}) = \int_{x^T\widetilde{A}x\geq \text{HAL}^2} \cdots \int \prod_{i=1}^{m} f_{i,\xi}(x_i)dx_1\cdots dx_m$$

$$\leq \int_{x^T\widetilde{A}x\geq \text{HAL}^2} \cdots \int \prod_{i=1}^{m} c_i\,[f_{b_i}(x_i) + f_{-b_i}(x_i)]\,dx_1\cdots dx_m,$$

with $\widetilde{A} = a_x a_x^T + a_y a_y^T$, $a_x = (a_{1,1},\ldots,a_{1,m})^T$, $a_y = (a_{2,1},\ldots,a_{2,m})^T$, where a_x is a vector formed from the first row of the matrix $A = (H^T\Sigma^{-1}H)^{-1}H^T\Sigma^{-1}$ (see equation (5)) and a_y is a vector formed from the second row of the matrix A.

Finally, taking into account the multi-binomial theorem, we get the following overbounding for the horizontal integrity risk

$$\mathbb{P}(\|Q_h\|_2 \geq \text{HAL}) \leq \sum_{j=1}^{2^m} \int_{x^T\widetilde{A}x\geq \text{HAL}^2} \cdots \int \prod_{i=1}^{m} c_i \cdot f_{(-1)^{\ell_i} b_i}(x_i)dx_1\cdots dx_m, \quad (27)$$

where the natural number $\ell_i = \ell_i(j) \in \mathbb{N}$ defines the sign of the mean $(-1)^{\ell_i} b_i$, $i = 1,\ldots,m$.

Let us define the set of vectors B_j

$$B_j = ((-1)^{\ell_1(j)}b_1,\ldots,(-1)^{\ell_m(j)}b_m)^T,\; j = 1,\ldots,2^m. \quad (28)$$

Putting together equations (22) and (27), we get the following overbounding for the horizontal integrity risk

$$\mathbb{P}(\|Q_h\|_2 \geq \text{HAL}) \leq \left[\prod_{i=1}^{m} c_i\right] \sum_{j=1}^{2^m} \left[1 - F_2\left(\text{HAL}^2, \overline{\Lambda}, \omega_j\right)\right], \quad (29)$$

where $\overline{\Lambda} = \text{diag}\{\lambda_{\max},\lambda_{\max}\}$ and $\omega_j = -\overline{\Lambda}^{-\frac{1}{2}} A_h B_j$ and the vector B_j, $j = 1,\ldots,2^m$ is defined in (28).

2) Excess-mass overbounding by using a single Gaussian PDF with adapted mean: Let us consider that the vector of biases $B = (b_1,\ldots,b_m)^T$ is such that $B \in \mathbb{B}$. It is assumed that the PDF $f_{i,\xi}(x)$ of the pseudo-range errors ξ_i, $i = 1,\ldots,m$, can be upper bounded by the PDF $f_{b_i}(x)$ of the Gaussian law $\mathcal{N}(b_i,\sigma_i^2)$ with the coefficient of inflation c_i :

$$f_{i,\xi}(x) \leq c_i f_{b_i}(x) \;\text{ for }\; x \in \mathbb{R}, \;\; i = 1,\ldots,m.$$

This method of overbounding is illustrated in Figure 7. By analogy with the previous method based on two Gaussian PDFs, the overbounding by using a single Gaussian PDF with adapted mean can be also interpreted as the method of “Excess-Mass PDF overbounding” proposed in [8].

In this case, the integrity risk is overbounded in the following manner

$$\mathbb{P}(\|Q_h\|_2 \geq \text{HAL}) = \int_{x^TAx\geq \text{HAL}^2} \cdots \int \prod_{i=1}^{m} f_{i,\xi}(x_i)dx_1\cdots dx_m$$

$$\leq \left[\prod_{i=1}^{m} c_i\right] \max_{B\in\mathbb{B}} \int_{x^TAx\geq \text{HAL}^2} \cdots \int \prod_{i=1}^{m} f_{b_i}(x_i)dx_1\cdots dx_m. \quad (30)$$

Putting together equations (20), (22) and (30), the following upper bound for the horizontal integrity risk is obtained

$$\mathbb{P}(\|Q_h\|_2 \geq \text{HAL}) \leq \left[\prod_{i=1}^{m} c_i\right] \max_{B\in\mathbb{B}} \left[1 - F_2\left(\text{HAL}^2, \overline{\Lambda}, \omega\right)\right] \quad (31)$$

with $\omega = -\overline{\Lambda}^{-\frac{1}{2}} A_h B$.

Finally, putting together equations (24) and (31), we get the formula which can be used for practical applications

$$\mathbb{P}(\|Q_h\|_2 \geq \text{HAL}) \leq \left[\prod_{i=1}^{m} c_i\right] \left[1 - F_2\left(\text{HAL}^2, \overline{\Lambda}, \omega\right)\right], \quad (32)$$

where $\omega = -\overline{\Lambda}^{-\frac{1}{2}} A_h B_j$, $j = \arg\max_{i=1,\ldots,2^m} \{\|A_h B_i\|_2\}$ and B_i is a vertex of the hyperrectangle $\mathbb{B}$, $i = 1,\ldots,2^m$. As previously, to reduce computational burden, the maximization of the Euclidean norm $\|A_h B\|_2$ over the set of 2^m vertices of the hyperrectangle $\mathbb{B}$ can be simplified by using the vector B_h^*, defined in (25), instead of $\overline{B}_h$. Hence, putting together (26) and (32), we get the simplified formula

$$\mathbb{P}(\|Q_h|_2 \geq \text{HAL}) \leq \left[\prod_{i=1}^{m} c_i\right] \left[1 - F_2\left(\text{HAL}^2, \overline{\Lambda}, \omega^*\right)\right], \quad (33)$$

where $\omega^* = -\overline{\Lambda}^{-\frac{1}{2}} B_h^*$.

6.3 Conservative bounds for the vertical instantaneous integrity risk (arbitrary distribution)

As we have mentioned in Section 5, the method of paired CDF overbounding (Paired Overbound Theorem 1) (see details in [5], [6]]) is well-adapted to the calculation of conservative bounds for the vertical instantaneous integrity risk because the LS estimation error Q_v is a linear combination of several independent pseudo-range errors $\xi_1, \ldots, \xi_m$, (see (17)). The conservative bounds for Q_v can be calculated for arbitrary lower and upper bounds of the CDF $F_{i,\xi}(x)$ of the i-th pseudo-range ξ_i

$$\underline{F}_{i,\xi}(x) \leq F_{i,\xi}(x) \leq \overline{F}_{i,\xi}(x) \quad \forall x \in \mathbb{R}.$$

The following paired CDF overbounding for the error vertical positioning $Q_v = \widehat{z} - z$:

$$\underline{F}_{\widehat{\xi}}(x) = \mathcal{N}\left(\mu, \sigma^2_{Q_v}\right) \text{ and } \overline{F}_{\widehat{\xi}}(x) = \mathcal{N}\left(-\mu, \sigma^2_{Q_v}\right),$$

where $\mu = \sum_{i=1}^{n} |a_{3,i}|\overline{b}_i$ and $\sigma^2_{Q_v} = \sum_{i=1}^{m} a^2_{3,i}\sigma^2_i$, will be used in the rest of the paper. This means that the instantaneous vertical integrity risk is upper bounded in the following manner:

$$\mathbb{P}\left(|\widehat{z} - z| \geq \text{VAL}\right) \leq 2\Phi\left(-\frac{\text{VAL} - \mu}{\sigma_{Q_v}}\right) \qquad (34)$$

7. CONSERVATIVE BOUNDS FOR THE INTEGRITY RISKS PER A GIVEN PERIOD OF TIME

7.1 Autoregressive first order model AR(1)

Experimental calculations show that the sequences of pseudo-range errors $\{\xi_{n,i}\}_{n\geq 1}$ are strongly auto-correlated and that the time series model can be approximated by AR(1) [12]. Due to the fact that the positioning errors $\widehat{x} - x$, $\widehat{y} - y$, and $\widehat{z} - z$ are represented as linear combinations of the pseudo-range errors $\xi_1, \ldots, \xi_m$ (see (7)), the horizontal and vertical positioning errors are also represented as an AR(1) process. It is assumed that the vector of pseudo-range errors is represented as an AR(1) process $\{\xi_n\}_{n\geq 1}$

$$\xi_n = (1-\lambda)\xi_{n-1} + \lambda\zeta_n, \;\; \xi_n = (\xi_{n,1}, \ldots, \xi_{n,m})^T, \qquad (35)$$

where $n = 1, 2, 3, \ldots$ is the current number of time step – GNSS epoch, $0 < 1 - \lambda < 1$ is the autoregressive coefficient, and $\{\zeta_n\}_{n\geq 1}$ is the i.i.d. vector innovation process. As it follows from (7), the vectors of horizontal and vertical positioning errors are given by $Q_{h,n} = A_h\xi_n$ and $Q_{v,n} = A_v\xi_n$, where A_h is a sub-matrix composed of the first two rows of the matrix A (see (7)) and A_v is a sub-matrix composed of the third row of the matrix A (see (7)). Hence,

$$\begin{aligned} Q_{h,n} &= (1-\lambda)Q_{h,n-1} + \lambda y_{h,n}, & (36)\\ Q_{v,n} &= (1-\lambda)Q_{v,n-1} + \lambda y_{v,n}, & (37)\end{aligned}$$

where the i.i.d. vectors $\{y_{h,n}\}_{n\geq 1}$ (resp. variables $\{y_{v,n}\}_{n\geq 1}$) obey a certain distribution $F_{h,y}$ (resp. $F_{v,y}$) , i.e., $y_{h,n} \sim F_{h,y}$ (resp. $y_{v,n} \sim F_{v,y}$).

7.2 Conservative bounds for the vertical integrity risk per a given period of time

The very first idea to calculate the probability that the random walk is absorbed by boundaries $-h$ and h by solving integral equations is due to [13], [14]. A pedagogical introduction to the first-passage-problem can be found in [15, Ch. 2]. The adaptation of general equations to the case of AR(1) process can be found in [16]. The recursive equation for the probability of the event $N_v = n$ is given by (see details in [17]) :

$$p_n(u) = \frac{1}{\lambda}\int_{-h}^{h} p_{n-1}(z) f_y\left(\frac{z-(1-\lambda)u}{\lambda}\right)dz, \qquad (38)$$

where $n = 2, 3, \ldots, T$, $p_n(u) = \mathbb{P}(N_v = n | Q_{v,0} = u)$, $f_y(x)$ is the PDF of y_n. The initial condition $p_1(u)$ is calculated in the following manner

$$\begin{aligned} p_1(u) &= \mathbb{P}\left(|(1-\lambda)u - \lambda y_1| > h\right)\\ &= 1 - F_y\left(\frac{h-(1-\lambda)u}{\lambda}\right) + F_y\left(\frac{-h-(1-\lambda)u}{\lambda}\right). \qquad (39)\end{aligned}$$

Finally, the probability of the event $\{1 \leq N_v \leq T\}$ provided that $Q_{v,0} = u$ is given by

$$\mathbb{P}(1 \leq N_v \leq T | Q_{v,0} = u) = \sum_{n=1}^{T} p_n(u). \qquad (40)$$

If the initial condition $Q_{v,0} = u$ is a random variable, we have to randomize the result in the following manner (under assumption that the distribution of initial state is known) :

$$\mathbb{P}(1 \leq N_v \leq T | u \in [-h, h]) = \frac{\int_{-h}^{h} f_{Q_{v,0}}(x) \sum_{n=1}^{T} p_n(x)dx}{\int_{-h}^{h} f_{Q_{v,0}}(x)dx}, \qquad (41)$$

where $u \sim F_{Q_{v,0}}$, $f_{Q_{v,0}}(x)$ is the PDF of $F_{Q_{v,0}}$.

Assumption 1: Let us assume that the CDF $F_y(x)$ of the innovation process $\{y_n\}_{n\geq 1}$ and the CDF $F_{Q_{v,0}}(x)$ of the initial state $Q_{v,0}$ obey the following inequality $\underline{F}_y(x) \leq F_y(x) \leq \overline{F}_y(x)$ and $\underline{F}_{Q_{v,0}}(x) \leq F_{Q_{v,0}}(x) \leq \overline{F}_{Q_{v,0}}(x)$ for $x \in \mathbb{R}$.

Lemma 1: Let us consider that Assumption 1 is satisfied. Then the upper bound $\overline{p}_n(u)$ for the probability $p_n(u)$ is given by

$$\begin{aligned} \overline{p}_n(u) \;=\; & \overline{p}_{n-1}(h)\overline{F}_y\left(\frac{h-(1-\lambda)u}{\lambda}\right)\\ & - \overline{p}_{n-1}(-h)\underline{F}_y\left(\frac{-h-(1-\lambda)u}{\lambda}\right)\\ & - \int_{-h}^{h} \underline{F}_y\left(\frac{z-(1-\lambda)u}{\lambda}\right)\mathbb{I}_{\{\overline{p}'_{n-1}(z)\geq 0\}}\overline{p}'_{n-1}(z)dz\\ & - \int_{-h}^{h} \overline{F}_y\left(\frac{z-(1-\lambda)u}{\lambda}\right)\mathbb{I}_{\{\overline{p}'_{n-1}(z)< 0\}}\overline{p}'_{n-1}(z)dz, \qquad (42)\end{aligned}$$

where $n = 2, 3, \ldots, T$, $\mathbb{I}_{\{A\}} = \begin{cases} 1 & \text{if A is true} \\ 0 & \text{if A is false}\end{cases}$ is the indicator function of the event A, $\overline{p}'_{n-1}(z) =$

$d\overline{p}_{n-1}(z)/dz$ and the upper bound for the probability $p_1(u)$ is given by

$$\overline{p}_1(u) = 1 - \underline{F}_y\left(\frac{h-(1-\lambda)u}{\lambda}\right) + \overline{F}_y\left(\frac{-h-(1-\lambda)u}{\lambda}\right). \tag{43}$$

Proposition 2: Let us consider that Assumption 1 is satisfied. Then the upper bound for the vertical integrity risk per a given period of time $p_r = \mathbb{P}(1 \le N_v \le T | u \in [-h,h])$ is given by

$$\begin{aligned} p_r \le \; & \frac{1}{a}\Big[\overline{p}_T(h)\overline{F}_{Q_{v,0}}(h) - \overline{p}_T(-h)\underline{F}_{Q_{v,0}}(-h) \\ & - \int_{-h}^{h} \underline{F}_{Q_{v,0}}(x)\mathbb{I}_{\{\overline{p}'_T(x)\ge 0\}}\overline{p}'_T(x)dx \\ & - \int_{-h}^{h} \overline{F}_{Q_{v,0}}(x)\mathbb{I}_{\{\overline{p}'_T(x)< 0\}}\overline{p}'_T(x)dx\Big], \end{aligned} \tag{44}$$

where $h = \text{VAL}$, $a = \underline{F}_{Q_{v,0}}(h) - \overline{F}_{Q_{v,0}}(-h)$, $\overline{p}_T(x) = \sum_{n=1}^{T}\overline{p}_n(x)$ and $\overline{p}'_T(x) = d\overline{p}_T(x)/dx$.

7.3 Conservative bounds for the horizontal integrity risk per a given period of time

The extension of the scalar first-passage-problem to the $\overline{m}$-dimensional vector AR(1), defined by equations (36), leads to the following recursive equation (for the sake of simplicity, only two-dimensional case, $\overline{m} = 2$, is considered in this section, see extension to $\overline{m} \ge 2$ in [17])

$$p_n(U) = \frac{1}{\lambda^2}\underset{\|Z\|_2\le h}{\int\cdots\int} p_{n-1}(Z) f_y\left(\frac{Z-(1-\lambda)U}{\lambda}\right)dZ, \tag{45}$$

where $n = 2,3,\ldots,T$, $p_n(U) = \mathbb{P}(N_h = n | Q_{h,0} = U)$, $f(X)$ is the PDF of y_n, $X, U, Z \in \mathbb{R}^2$, $dZ = dz_1 dz_2$. The initial condition $p_1(U)$ is calculated in the following manner

$$\begin{aligned} p_1(U) &= \mathbb{P}(\|(1-\lambda)U - \lambda y_1\|_2 > h) \\ &= \frac{1}{\lambda^2}\underset{\|Z\|_2\ge h}{\int\cdots\int} f_y\left(\frac{Z-(1-\lambda)U}{\lambda}\right)dZ. \end{aligned} \tag{46}$$

The probability of the event $\{1 \le N_h \le T\}$ provided that $Q_{h,0} = U$ is given by previously defined equation (40) and the probability $\mathbb{P}(1 \le N_h \le T | u \in [-h,h])$ by previously defined equation (41).

Assumption 2: Let us assume that the CDF $F_y(X) = F_{y,1}(x_1)F_{y,2}(x_2)$ of the innovation process $\{y_n\}_{n\ge 1}$ and the CDF $F_{Q_{h,0}}(X) = F_{Q_{h,0},1}(x_1)F_{Q_{h,0},2}(x_2)$ of the initial state $Q_{h,0}$ obey the following inequality $\underline{F}_{y,i}(x) \le F_{y,i}(x) \le \overline{F}_{y,i}(x)$ and $\underline{F}_{Q_{h,0},i}(x) \le F_{Q_{h,0},i}(x) \le \overline{F}_{Q_0,i}(x)$ for $x \in \mathbb{R}$, where $i = 1,2$.

Lemma 2: Let us consider that Assumption 2 is satisfied. Then the upper bound for the probability $p_n(U)$ is given by Lemma 1, where z is replaced with z_1, u with u_2, and the function $\overline{p}_{n-1}(z)$ is replaced with the function

$$\begin{aligned} &\overline{I}(z_1,u_2) = \\ &= \overline{p}_{n-1}\left(z_1, \sqrt{h^2-z_1^2}\right)\overline{F}_{y,2}\left(\frac{\sqrt{h^2-z_1^2}-(1-\lambda)u_2}{\lambda}\right) \\ &- \overline{p}_{n-1}\left(z_1, -\sqrt{h^2-z_1^2}\right)\underline{F}_{y,2}\left(\frac{-\sqrt{h^2-z_1^2}-(1-\lambda)u_2}{\lambda}\right) \\ &- \int_{-\sqrt{h^2-z_1^2}}^{\sqrt{h^2-z_1^2}} \underline{F}_{y,2}\left(\frac{z_2-(1-\lambda)u_2}{\lambda}\right)p^+(z_1,z_2)dz_2 \\ &- \int_{-\sqrt{h^2-z_1^2}}^{\sqrt{h^2-z_1^2}} \overline{F}_{y,2}\left(\frac{z_2-(1-\lambda)u_2}{\lambda}\right)p^-(z_1,z_2)dz_2, \end{aligned}$$

for $-h < z_1 < h$, $\overline{I}(-h,u_2) = \overline{I}(h,u_2) = 0$, $n = 2,3,\ldots,T$, $p^+(z_1,z_2) = \mathbb{I}_{\{\overline{p}'_{n-1}(z_1,z_2)\ge 0\}}\overline{p}'_{n-1}(z_1,z_2)$, $p^-(z_1,z_2) = \mathbb{I}_{\{\overline{p}'_{n-1}(z_1,z_2)< 0\}}\overline{p}'_{n-1}(z_1,z_2)$ and $\overline{p}'_{n-1}(z_1,z_2) = \dfrac{\partial\overline{p}_{n-1}(z_1,z_2)}{\partial z_2}$. The upper bound $\overline{p}_1(U)$ for the probability $p_1(U)$ is given by

$$\begin{aligned} \overline{p}_1(U) = 1 &+ \int_{-h}^{h}\overline{F}_{y,1}\left(\frac{z_1-(1-\lambda)u_1}{\lambda}\right)I^+(z_1,u_2)dz_1 \\ &+ \int_{-h}^{h}\underline{F}_{y,1}\left(\frac{z_1-(1-\lambda)u_1}{\lambda}\right)I^-(z_1,u_2)dz_1, \end{aligned}$$

where $I^+(z_1,u_2) = \mathbb{I}_{\{\underline{I}'_1(z_1,u_2)\ge 0\}}\underline{I}'_1(z_1,u_2)$, $I^-(z_1,u_2) = \mathbb{I}_{\{\underline{I}'_1(z_1,u_2)< 0\}}\underline{I}'_1(z_1,u_2)$

$$\begin{aligned} \underline{I}_1(z_1,u_2) = \max\Bigg\{ & \underline{F}_{y,2}\left(\frac{\sqrt{h^2-z_1^2}-(1-\lambda)u_2}{\lambda}\right) \\ & -\overline{F}_{y,2}\left(\frac{-\sqrt{h^2-z_1^2}-(1-\lambda)u_2}{\lambda}\right), 0\Bigg\} \end{aligned}$$

and $\underline{I}'_1(z_1,u_2) = \dfrac{\partial \underline{I}_1(z_1,u_2)}{\partial z_1}$.

Proposition 3: Let us consider that Assumption 2 is satisfied. Then the upper bound for the horizontal integrity risk per a given period of time $p_r = \mathbb{P}(1 \le N_h \le T | \|U\|_2 < h)$ is given by

$$\begin{aligned} p_r \le \; & \frac{1}{a}\Big[-\int_{-h}^{h}\underline{F}_{Q_0,1}(x_1)\mathbb{I}_{\{I'_0(x_1)\ge 0\}}I'_0(x_1)dx_1 \\ & - \int_{-h}^{h}\overline{F}_{Q_0,1}(x_1)\mathbb{I}_{\{I'_0(x_1)< 0\}}I'_0(x_1)dx_1\Big], \end{aligned}$$

where $h = \text{HAL}$, $I_0(-h) = I_0(h) = 0$ and

$$\begin{aligned} I_0(x_1) &= \overline{p}_T\left(x_1, \sqrt{h^2-x_1^2}\right)\overline{F}_{Q_0,2}\left(\sqrt{h^2-x_1^2}\right) \\ &- \overline{p}_T\left(x_1, -\sqrt{h^2-x_1^2}\right)\underline{F}_{Q_0,2}\left(-\sqrt{h^2-x_1^2}\right) \\ &- \int_{-\sqrt{h^2-x_1^2}}^{\sqrt{h^2-x_1^2}}\underline{F}_{Q_0,2}(x_2)\mathbb{I}_{\{\overline{p}'_T(x_1,x_2)\ge 0\}}\overline{p}'_T(x_1,x_2)dx_2 \\ &- \int_{-\sqrt{h^2-x_1^2}}^{\sqrt{h^2-x_1^2}}\overline{F}_{Q_0,2}(x_2)\mathbb{I}_{\{\overline{p}'_T(x_1,x_2)< 0\}}\overline{p}'_T(x_1,x_2)dx_2, \end{aligned}$$

for $-h < x_1 < h$, $I'_0(x_1) = \dfrac{dI'_0(x_1)}{dx_1}$, $\overline{p}_T(X) = \sum_{n=1}^{T} \overline{p}_n(X)$, $\overline{p}'_T(x_1, x_2) = \dfrac{\partial \overline{p}_T(x_1, x_2)}{\partial x_2}$. The constant a is given by

$$a = -\int_{-h}^{h} \overline{F}_{Q_0,1}(x_1)\mathbb{I}_{\{I'(x_1)\geq 0\}} I'(x_1)dx_1 - \int_{-h}^{h} \underline{F}_{Q_0,1}(x_1)\mathbb{I}_{\{I'(x_1)<0\}} I'(x_1)dx_1,$$

$I'(x_1) = \dfrac{dI(x_1)}{dx_1}$ and $I(x_1) = \max\left\{\underline{F}_{Q_0,2}\left(\sqrt{h^2 - x_1^2}\right) - \overline{F}_{Q_0,2}\left(-\sqrt{h^2 - x_1^2}\right), 0\right\}$. As a final remark, it should be noted that the above mentioned equations can produce aberrant results for an unusual choice of relations between the CDF overbounds (see Assumptions 1 and 2) and the parameters λ and h.

7.4 Using the"excess-mass PDF overbounding"

Sometimes it is necessary to use the bounds for the PDF of Q_0 and/or for the PDF of y_n. Such kind of bounds are usually used to overbound the distributions with excess-mass functions (see [8]).

Conservative bounds for both horizontal and vertical integrity risk per a given period of time are considered now. For this reason, it is assumed that the dimension of the vector AR(1) process is $\overline{m} \geq 1$.

Assumption 3: Let us assume that the PDF $f_y(X)$ of the innovation process $\{y_n\}_{n\geq 1}$ and the PDF $f_{Q_0}(X)$ of the initial state Q_0 obey the following inequality

$$f_y(X) \leq \overline{f}_y(X) \text{ and } f_{Q_0}(X) \leq \overline{f}_{Q_0}(X) \text{ for } X \in \mathbb{R}^{\overline{m}}.$$

Let us assume that Assumption 3 is satisfied. Then the above-mentioned recursive equations have to be replaced with the following inequalities

$$p_n(U) \leq \frac{1}{\lambda^{\overline{m}}} \int \cdots \int_{\|Z\|_2 \leq h} p_{n-1}(Z)\overline{f}_y\left(\frac{Z - (1-\lambda)U}{\lambda}\right) dZ, \tag{47}$$

$n = 2, 3, \dots, T$, and the initial condition $p_1(U)$ is also upper bounded in the following manner

$$p_1(U) \leq \frac{1}{\lambda^{\overline{m}}} \int \cdots \int_{\|Z\|_2 \geq h} \overline{f}_y\left(\frac{Z - (1-\lambda)U}{\lambda}\right) dZ. \tag{48}$$

The threshold h is equal to HAL or VAL. This method of the risk overbounding can lead to very conservative results due to the recursive character of the above-mentioned equations. This problem can be especially important for large values of T. Hence, a special attention should be paid to the choice of the upper bounds $\overline{f}_y(X)$ and $\overline{f}_{Q_0}(X)$.

8. NUMERICAL EXAMPLES

The first example is devoted to the conservative bounds for the horizontal/vertical instantaneous integrity risk. Let us consider the following scenario : HAL $= 40$ m, VAL $= 35$ m, the GPS constellation is simulated with the YUMA almanac, week 0593 (Jan. 2011), available at http://celestrak.com/. It is assumed that the diagonal variance-covariance matrix of the pseudo-range noise is $\Sigma = \mathrm{diag}\{4, \dots, 4\}\,\mathrm{m}^2$ and the pseudo-range biases b_i are bounded by $\overline{b}_i = 3$ m, $i = 1, \dots, m$. The geographic coordinates of the user are $(\phi, \lambda, h) = (48°16'7'', 4°3'57'', 178\,\mathrm{m})$ and the elevation mask angle is set to $7°$. The major radius of the horizontal error

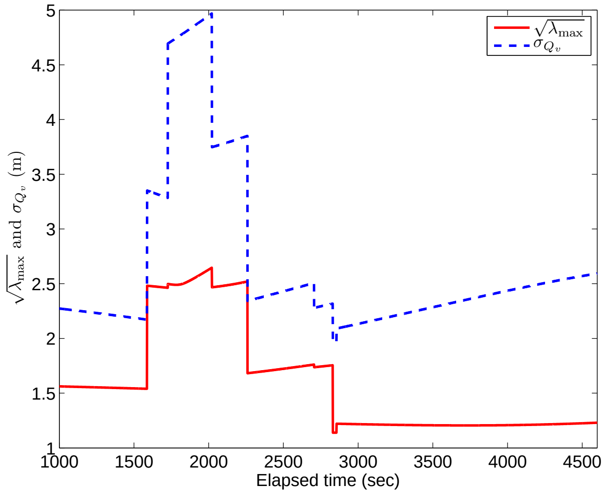

Fig. 8. The major radius of the horizontal error ellipse $\sqrt{\lambda_{\max}}$ and the SD σ_{Q_v} of the vertical error Q_v as functions of the elapsed time.

ellipse $\sqrt{\lambda_{\max}}$ and the standard deviation (SD) σ_{Q_v} of the vertical error Q_v as functions of the elapsed time are shown in Figure 8. The conservative worst case horizontal bias $\|B_h^*\|_2$, the worst case horizontal bias $\left\|\overline{B}_h\right\|_2$ and the worst case vertical bias μ as functions of the elapsed time are shown in Figure 9. Finally, the conservative bounds for the horizontal/vertical instantaneous integrity risk are presented in Figure 10. The conservative bound for the horizontal instantaneous integrity risk is calculated by using (32) with the inflation coefficient $c_i = 3$, $i = 1, \dots, m$. The conservative bound for the vertical instantaneous integrity risk is calculated by using (34).

The second example is devoted to Proposition 2 with the paired CDF overbounding of y_n and Q_0. A special method of numerical integration based on the Gaussian quadrature and the 5-point numerical derivative has been designed to calculate the conservative bounds for the integrity risk per a given period of time. Let us consider the following scenario : VAL $= 25$ m, $T = 150$ sec. It is assumed that $\sigma_{Q_v}^2 = \mathrm{Var}\,Q_{v,n} = 12\,\mathrm{m}^2$ and that Assumption 1 is satisfied with the following bounds for the CDF $F_y(x)$ of the innovation process $\{y_n\}_{n\geq 1}$ and

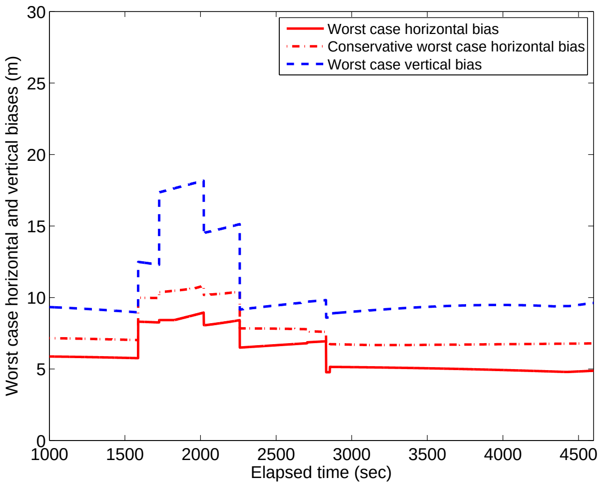

Fig. 9. The conservative worst case horizontal bias $\|B_h^*\|_2$, the worst case horizontal bias $\|\overline{B}_h\|_2$ and the worst case vertical bias μ as functions of the elapsed time.

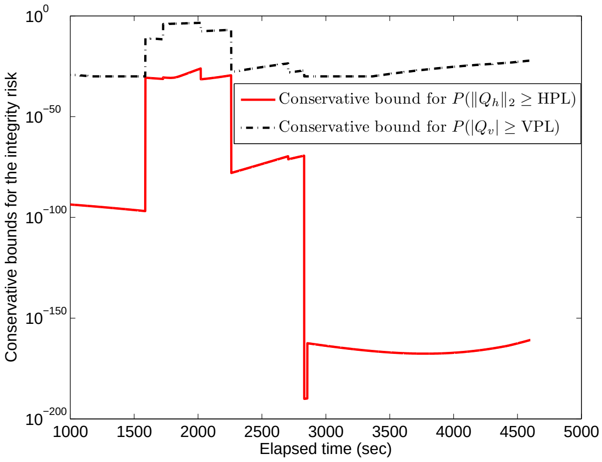

Fig. 10. The conservative bounds for the horizontal/vertical instantaneous integrity risk.

the CDF $F_{Q_{v,0}}(x)$ of the initial state $Q_{v,0}$:

$$\underline{F}_y(x) = \mathcal{N}\left(\mu,\sigma_y^2\right) \le F_y(x) \le \overline{F}_y(x) = \mathcal{N}\left(-\mu,\sigma_y^2\right)$$

and

$$\underline{F}_{Q_{v,0}}(x) = \mathcal{N}(\mu,\sigma^2_{Q_v}) \le F_{Q_{v,0}}(x) \le \overline{F}_{Q_{v,0}}(x) = \mathcal{N}(-\mu,\sigma^2_{Q_v}),$$

where $\mu = 8$ m, $\sigma_y^2 = \frac{1-(1-\lambda)^2}{\lambda^2}\sigma^2_{Q_v}$ and $1-\lambda \in [0, 0.9]$.

The stopping boundary h in (12) is set to VAL. Let us now compare the vertical integrity risk $\mathbb{P}(1 \le N_v \le T | u \in [-h, h])$ defined by equation (41) for the Gaussian innovation process and the initial state with the "worst case expectation", i.e., $\mu = 8$ m, with the upper bound of this risk given by Proposition 2 for unknown distributions of the innovation process and initial state. The risk and its conservative overbound as functions of $1-\lambda$ are presented in Figure 11. The risk that the absorption occurs at one of the barriers $-h = -\text{VAL}$ or $h = \text{VAL}$ at or before the 150-th step for the known Gaussian AR(1) process with $\mu = 8$ m is shown in dashed line and the its overbound for the AR(1) process with unknown distributions is shown in solid line in Figure 11.

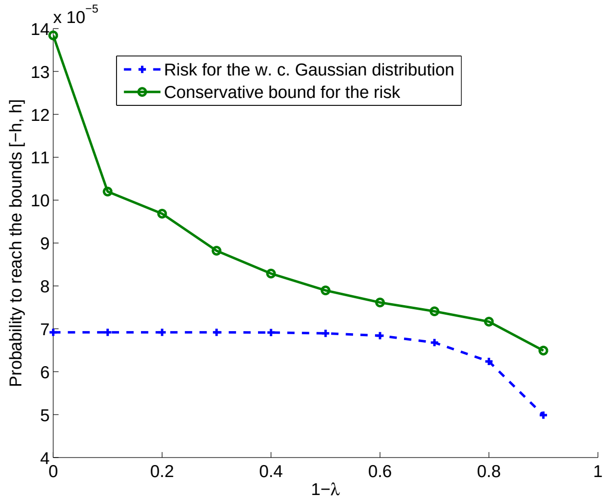

Fig. 11. The conservative bound of the vertical integrity risk per a given period of time and the vertical integrity risk calculated for the worst case (w. c.) Gaussian AR(1) process.

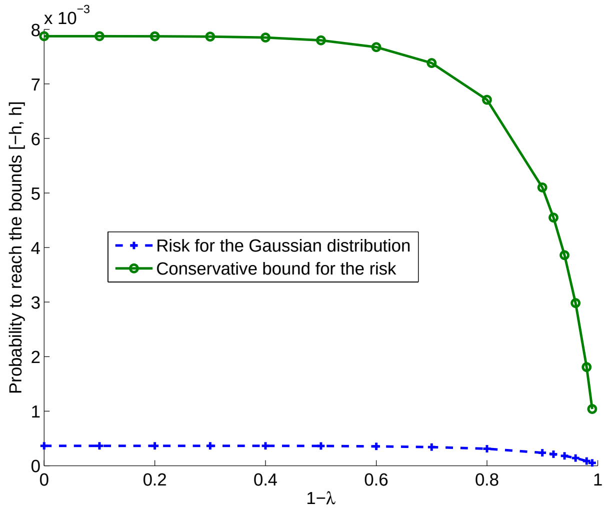

Fig. 12. The conservative bound of the vertical integrity risk per a given period of time by using the excess-mass PDF overbounding and the vertical integrity risk calculated for the Gaussian AR(1) process.

The third example is devoted to the usage of the excess-mass PDF overbounding of y_n. Let us consider the following scenario : VAL $= 25$ m, $T = 50$ sec. The scalar AR(1) process is given by equation (37) with the autoregressive coefficient $1-\lambda \in [0, 0.99]$. It is assumed that Assumption 3 is satisfied with the following excess-mass overbound for the PDF $f_y(x)$ of the innovation process $\{y_n\}_{n\geq 1}$:

$$f_y(x) \leq \overline{f}_y(x) = c \cdot f(x), \quad (49)$$

where $f(x)$ is the PDF of the Gaussian distribution $\mathcal{N}\left(\mu, \frac{1-(1-\lambda)^2}{\lambda^2}\sigma^2_{Q_v}\right)$, $\mu = 10$ m, $\sigma^2_{Q_v} = \mathrm{Var}\, Q_{v,n} = 12\,\mathrm{m}^2$ and $c = 1.1$. The stopping boundary h in (12) is set to VAL. The comparison between the Gaussian AR(1) process and the AR(1) process with the innovation PDF overbounded with the excess-mass function is presented in Figures 12. It follows from Figures 11 and 12 that the conservative bound for the vertical integrity risk per a given period of time obtained by using the excess-mass PDF overbounding is more conservative that the same bound obtained by using the paired CDF overbounding.

9. CONCLUSION

This paper addresses the problem of horizontal/vertical integrity risk overbounding. Two new methods of the instantaneous (i.e., per one GNSS measurement) horizontal integrity risk overbounding are proposed. These methods provide the users with a conservative estimation of the probability that the horizontal error is greater than a prescribed horizontal protection level. The calculation of conservative bounds for the horizontal/vertical integrity risk per a given period of time is reduced to the first-passage-problem for the autoregressive process. A numerical method based on the integral equations has been proposed to find a conservative bound for the probability that the autoregressive process absorption at the barrier occurs at or before a given period of time.

The theoretical findings proposed in the paper represent a mathematical background for the vertical and horizontal integrity risk overbounding in exact terms following the requirements of the MOPS for GPS/Galileo.

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REFERENCES

[1] B. Parkinson and J. Spilker, Eds., *Global Positioning System : Theory and Applications, Volume 1*. American Institue of Aeronautics and Astronautics, 1996.

[2] A. A. Borovkov, *Mathematical Statistics*. Amsterdam, NL: Gordon and Breach Sciences Publishers, 1998.

[3] K.-R. Koch, *Parameter Estimation and Hypothesis Testing in Linear Models*. New York, NY, USA: Springer-Verlag New York, Inc., 1988.

[4] B. DeCleene, "Defining Pseudorange Integrity - Overbounding," in *Proceedings of The 13th International Meeting of The Satellite Division of the Institute of Navigation. Salt Lake City, UT, USA*, 19-22 September 2000, pp. 1916–1924.

[5] J. Rife, S. Pullen, B. Pervan, and P. Enge, "Paired Overbounding and Application to GPS Augmentation," in *Proceedings of The Position Location and Navigation Symposium. PLANS 2004, USA*, 26-29 April 2004, pp. 439–446.

[6] J. Rife, S. Pullen, P. Enge, and B. Pervan, "Paired Overbounding for Nonideal LAAS and WAAS Error Distributions," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 42, no. 4, pp. 1386 – 1395, october 2006.

[7] T. Walter, J. Blanch, and P. Enge, "L5 Satellite Based Augmentation Systems Protection Level Equations," in *Proceedings of the International Global Navigation Satellite Systems Society IGNSS Symposium 2007, Sydney, Australia*, 4-6 December 2007.

[8] J. Rife, T. Walter, and J. Blanch, " Overbounding SBAS and GBAS Error Distributions with Excess-Mass Functions," in *Proceedings of the GNSS 2004 Internat. Symp. On GNSS/GPS, Sydney, Australia*, 6-8 December 2004.

[9] T. Walter, J. Blanch, and J. Rife, "Treatment of Biased Error Distributions in SBAS," *Journal of Global Positioning Systems*, vol. 3, no. 1-2, pp. 265–272, 2004.

[10] J. Rife and S. Pullen, "The Impact of Measurement Biases on Availability for CAT III LAAS," *Navigation, the Journal of the Institute of Navigation*, vol. 52, no. 4, pp. 215–228, 2005.

[11] N. Johnson, S. Kotz, and N. Balakrishnan, *Continuous univariate distributions, vol. 2*, ser. Wiley series in probability and mathematical statistics: Applied probability and statistics. Wiley & Sons, 1995.

[12] I. Nikiforov and Roturier B., " Advanced RAIM Algorithms: First Results," in *Proceedings of The 18th International Meeting of The Satellite Division of the Institute of Navigation. Long-Beach, CA, USA*, 13-16 Septembre 2005.

[13] J. H. B. Kemperman, *The General One-dimensional Random Walk with Absorbing Barriers: with Applications to Sequential Analysis*. Amsterdam, NL: 's-Gravenhage, 1951.

[14] E. S. Page, "Continuous inspection schemes," *Biometrika*, vol. 41, no. 1–2, pp. 100–114, Jun. 1954.

[15] D. Cox and H. Miller, *The Theory of Stochastic Processes*. Taylor & Francis, 1977. [Online]. Available: https://books.google.fr/books?id=NeR5JEunGYwC

[16] S. V. Crowder, "A simple method for studying run-length distributions of exponentially weighted moving average charts," *Technometrics*, vol. 29, no. 4, pp. pp. 401–407, 1987.

[17] I. Nikiforov, "Bounding the risk probability," in *Proceedings of 20th International Conference, DCCN 2017, September 25-29, Moscow, Russia*, vol. 700, Series: Communications in Computer and Information Science. Springer, 2017, pp. 135–145.